DOI 10.12851/EESJ201502C06ART01

Victor I. Shulga,
ScD, associate professor,

Igor Bilan,
expert mathematician,
East European University of
Economics and Business Administration

Obtaining an Approximate Solution of Mathematical Physics Equations of Several Variables by Means of Taylor Series

Key words: *Taylor, mathematical physics, several variables.*

Annotation: *The aim of this paper is to prove the following obtaining an approximate solution of mathematical physics equations of several variables by means of Taylor series. For example use 2 variables and the Dirichlet boundary condition. Easily generalized to 3,4,5,6,… variables and the Robin boundary condition and the Dirichlet boundary condition.*

Abstract. The aim of this paper is to prove the following obtaining an approximate solution of mathematical physics equations of several variables by means of Taylor series. For example use 2 variables and the Dirichlet boundary condition. Easily generalized to 3,4,5,6,… variables and the Robin boundary condition and the Dirichlet boundary condition.

Definition 1. For example

$f_1(x_1,x_2)\frac{\partial^2\Phi(x_1,x_2)}{\partial x_1\partial x_1} + f_2(x_1,x_2)\frac{\partial^2\Phi(x_1,x_2)}{\partial x_2\partial x_2} + f_3(x_1,x_2)\frac{\partial^2\Phi(x_1,x_2)}{\partial x_1\partial x_2} + f_4(x_1,x_2)\frac{\partial^2\Phi(x_1,x_2)}{\partial x_1} +$
$f_5(x_1,x_2)\frac{\partial^2\Phi(x_1,x_2)}{\partial x_2} + f_6(x_1,x_2) = 0$ on Ω is the equation of mathematical physics.

Definition 2. The Dirichlet boundary condition is $\Phi(x_1,x_2) = g(x_1,x_2)$ on $\partial\Omega$.

Definition 3. $f_i(x_{1,0}+h_1,x_{2,0}+h_2) = \sum_{k_1+k_2\le n_1} W_{i,k_1k_2}h^{k_1k_2} + R_{n_1}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_1}(x_{1,0}+h_1,x_{2,0}+h_2) \approx 0$ on Ω.

Definition 4. $\Phi(x_{1,0}+h_1,x_{2,0}+h_2) = \sum_{k_1+k_2\le n_1} W_{7,k_1k_2}h^{k_1k_2} + R_{n_2}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_2}(x_{1,0}+h_1,x_{2,0}+h_2) \approx 0$ on Ω.

Definition 5. $0 = \sum_{k_1+k_2\le n_1} 0h^{k_1k_2} + R_{n_3}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_3}(x_{1,0}+h_1,x_{2,0}+h_2) = 0$ on Ω.

Definition 6.
$f_1(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1\partial h_1} = \sum_{k_1+k_2\le n_1} W_{8,k_1k_2}h^{k_1k_2} + R_{n_4}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_4}(x_{1,0}+h_1,x_{2,0}+h_2) \approx 0$ on Ω.

Definition 7.
$f_2(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2\partial h_2} = \sum_{k_1+k_2\le n_1} W_{9,k_1k_2}h^{k_1k_2} + R_{n_5}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_5}(x_{1,0}+h_1,x_{2,0}+h_2)\approx 0$ on Ω.

Definition 8.
$f_3(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1\partial h_2} = \sum_{k_1+k_2\le n_1} W_{10,k_1k_2}h^{k_1k_2} + R_{n_6}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_6}(x_{1,0}+h_1,x_{2,0}+h_2)\approx 0$ on Ω.

Definition 9.
$f_4(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1} = \sum_{k_1+k_2\le n_1} W_{11,k_1k_2}h^{k_1k_2} + R_{n_7}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_7}(x_{1,0}+h_1,x_{2,0}+h_2)\approx 0$ on Ω.

Definition 10.
$f_5(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2} = \sum_{k_1+k_2\le n_1} W_{12,k_1k_2}h^{k_1k_2} + R_{n_8}(x_{1,0}+h_1,x_{2,0}+h_2)$ and $R_{n_8}(x_{1,0}+h_1,x_{2,0}+h_2)\approx 0$ on Ω.

Theorem 11. *Using the definitions of this article, we get coefficients of Taylor series of* $\Phi(x_{1,0}+h_1,x_{2,0}+h_2)\approx\sum_{k_1+k_2\le n_1} W_{7,k_1k_2}h^{k_1k_2}$ *as the solution of a system of linear equations.*

Proof. Let W_{7,k_1k_2} be unknown variables $y_{k_1k_2}$. Let l be a maximum of n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8. Let l_1 and l_2 be $l_1+l_2\le l$ and $0\le l_1$ and $0\le l_2$. Consider a coefficient of $h_1^{l_1}h_2^{l_2}$.
The coefficient of $h_1^{l_1}h_2^{l_2}$ in $f_1(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1\partial h_1}$ is

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{1,k_{1,1},k_{2,1}}(k_{1,2}+2)(k_{1,2}+1)y_{k_{1,2}+2,k_{2,2}}$.

The coefficient of $h_1^{l_1}h_2^{l_2}$ in $f_2(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2\partial h_2}$ is

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{2,k_{1,1},k_{2,1}}(k_{2,2}+2)(k_{2,2}+1)y_{k_{1,2},k_{2,2}+2}$.

The coefficient of $h_1^{l_1}h_2^{l_2}$ in $f_3(x_{1,0}+h_1,x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1\partial h_2}$ is

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{3,k_{1,1},k_{2,1}}(k_{1,2}+1)(k_{2,2}+1)y_{k_{1,2}+1,k_{2,2}+1}$.

The coefficient of $h_1^{l_1} h_2^{l_2}$ in $f_4(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1}$ is

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{4,k_{1,1},k_{2,1}}(k_{1,2}+1)y_{k_{1,2}+1,k_{2,2}}$.

The coefficient of $h_1^{l_1} h_2^{l_2}$ in $f_5(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2}$ is

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{5,k_{1,1},k_{2,1}}(k_{2,2}+1)y_{k_{1,2},k_{2,2}+1}$.

The coefficient of $h_1^{l_1} h_2^{l_2}$ in $f_6(x_{1,0}+h_1, x_{2,0}+h_2)$ is $W_{6,l_1 l_2}$. The coefficient of $h_1^{l_1} h_2^{l_2}$ in 0 is 0.

We get the coefficient of $h_1^{l_1} h_2^{l_2}$ in $f_6(x_{1,0}+h_1, x_{2,0}+h_2) + f_5(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2} + f_4(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1} + f_3(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1 \partial h_2} + f_2(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_2 \partial h_2} + f_1(x_{1,0}+h_1, x_{2,0}+h_2)\frac{\partial^2\Phi(x_{1,0}+h_1,x_{2,0}+h_2)}{\partial h_1 \partial h_1}$ is

$W_{6,l_1 l_2}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{5,k_{1,1},k_{2,1}}(k_{2,2}+1)y_{k_{1,2},k_{2,2}+1}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{4,k_{1,1},k_{2,1}}(k_{1,2}+1)y_{k_{1,2}+1,k_{2,2}}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{3,k_{1,1},k_{2,1}}(k_{1,2}+1)(k_{2,2}+1)y_{k_{1,2}+1,k_{2,2}+1}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{2,k_{1,1},k_{2,1}}(k_{2,2}+2)(k_{2,2}+1)y_{k_{1,2},k_{2,2}+2}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{1,k_{1,1},k_{2,1}}(k_{1,2}+2)(k_{1,2}+1)y_{k_{1,2}+2,k_{2,2}}$

and using the definitions 1, 5 we get $W_{6,l_1 l_2}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{5,k_{1,1},k_{2,1}}(k_{2,2}+1)y_{k_{1,2},k_{2,2}+1}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{4,k_{1,1},k_{2,1}}(k_{1,2}+1)y_{k_{1,2}+1,k_{2,2}}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{3,k_{1,1},k_{2,1}}(k_{1,2}+1)(k_{2,2}+1)y_{k_{1,2}+1,k_{2,2}+1}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{2,k_{1,1},k_{2,1}}(k_{2,2}+2)(k_{2,2}+1)y_{k_{1,2},k_{2,2}+2}$ +

$$\sum_{0\le k_{1,1}\le l_1;\ and\ 0\le k_{2,1}\le l_2;\ and\ 0\le k_{1,2}\le l_1;\ and\ 0\le k_{2,2}\le l_2;\ and\ k_{1,1}+k_{1,2}=l_1;\ and\ k_{2,1}+k_{2,2}=l_2}$$

$W_{1,k_{1,1},k_{2,1}}(k_{1,2}+2)(k_{1,2}+1)y_{k_{1,2}+2,k_{2,2}}$ =0

We have $\frac{(l+1)(l+2)}{2}$ linear equations and $\frac{(l+3)(l+4)}{2}$ unknown variables. Using the Dirichlet boundary condition we get $\frac{(l+3)(l+4)}{2}-\frac{(l+1)(l+2)}{2}$ linear equations. If $\{x_{1,s},x_{2,s}\}_{s=1}^{s=\frac{(l+3)(l+4)}{2}-\frac{(l+1)(l+2)}{2}} \in \partial\Omega$ then we have $\Phi(x_{1,s},x_{2,s}) = \sum_{k_1+k_2\le l+2} y_{k_1,k_2}(x_{1,s}-x_{1,0})^{k_1}(x_{2,s}-x_{2,0})^{k_2} = g(x_{1,s},x_{2,s})$.

We obtain $\frac{(l+3)(l+4)}{2}$ the unknown variables and $\frac{(l+3)(l+4)}{2}$ the linear equations. Find y_{k_1,k_2} . Finally, we obtain $\Phi(x_{1,0}+h_1,x_{2,0}+h_2) \approx \sum_{k_1+k_2\le l+2} y_{k_1,k_2} h^{k_1 k_2}$.

Theorem 12. *Easily generalized to 3,4,5,6,… variables and the Robin boundary condition.*

Proof. The proof is left to the reader.

This method is applicable to the majority of the equations of mathematical physics, but, if system of linear equations is well-posed problem, then we have the approximate solution of the equation of mathematical physics, else the method is not suitable.