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Application of Fractional Steps Method for Determining the Stress and Strain Field during the Temperature Load of Cylindrical Bodies

Key words: *temperature field, stress stain, heat stroke.*

Annotation: *The problem of determining the and stress stain state in the hollow cylindrical body with heat stroke, caused by the instantaneous change of temperature of the part of inner surface is considered.*

Introduction. Constructions of cylindrical shape are widely used in aviation, mining, oil, gas industry, thermal engineering, building and other branches of engineering. Constructions can be influenced by the complex unsteady pressure and temperature load during the process of production and implementation, so finding the fields of stress and strain at various loads is one of important tasks for modeling the behavior of elastic and cylindrical bodies. Uneven heating of bodies in combination with power loads leads to the complex deformation processes. For modeling the behavior of such construction elements there should be defined unsteady thermo-mechanical condition both on the elastic stage of deformation, and by the elastic limit. The existing numerical methods for solving such non-stationary tasks lead, as a rule, to large computational difficulties associated with the solution of algebraic systems of equations, which are not always effective.

Currently, such problems are solved by numerical methods: finite element method, finite difference method, and others.

The application of methods that simplify the calculation and have a high accuracy of results during finding the stress and strain of the cylindrical bodies in the case of temperature load during cooling the body is relevant.

Statement of the problem. Temperature field for the isotropic body in the case of heat accounting, which is produced during cyclic deformation process, is determined by solving the unsteady heat equation under certain initial and boundary conditions.

$$\frac{\partial T}{\partial t}=\frac{a}{H_1H_2H_3}\left\{\frac{\partial}{\partial\alpha^1}\left(\frac{H_2H_3}{H_1}\cdot\frac{\partial T}{\partial\alpha^1}\right)+\frac{\partial}{\partial\alpha^2}\left(\frac{H_1H_3}{H_2}\cdot\frac{\partial T}{\partial\alpha^2}\right)+\frac{\partial}{\partial\alpha^3}\left(\frac{H_2H_1}{H_3}\frac{\partial T}{\partial\alpha^3}\right)+\frac{W_\bullet}{\lambda}\right\},(1)$$

where $W_\bullet$ is the scattering function, H_i - Lame parameters (i= 1, 2, 3), derivatives by time are designated with point

$$W_\bullet=S_{ij}\dot{Э}_{ij}-\frac{1}{2G}S_{ij}\dot{S}_{ij}+\frac{\sigma_{ii}}{3}\left(\dot{\varepsilon}_{jj}-3\alpha_T\frac{\partial T}{\partial t}\right)-\frac{\sigma_{ii}}{3K}\dot{\sigma}_{jj}$$

$$S_{ij}=\sigma_{ij}-\delta_{ij}\sigma,\ Э_{ij}=\varepsilon_{ij}-\delta_{ij}\varepsilon \qquad (2)$$

$$\sigma=\frac{\sigma_{ii}}{3},\varepsilon=\frac{\varepsilon_{ii}}{3},\ G=\frac{E}{2(1+\nu)},\ K=\frac{3E}{1-2\nu}$$

Here S_{ij} ,$Э_{ij}$ are deviators of tensors of stress and strain, σ_{ij} ,ε_{ij} are the tensors of stress and strain.
The initial temperature distribution in the body that corresponds to the natural relaxed state of the body is set in such a way:

$$T=T_0(\alpha^i) \text{ at } t=0. \qquad (3)$$

The boundary conditions reflecting the influence of environment on body temperature, are defined as follows:

$$\lambda\cdot\frac{\partial T}{\partial n}=-\alpha(T-\theta)-q, \qquad (4)$$

where n is the external norm to the surface of the body, α_T is the coefficient of linear thermal expansion, α is the coefficient of heat transfer, Θ is the temperature of environment, q is the heat flux.

In general case of the values α, Θ, q. can depend on time and position of point (α^1, α^2, $\alpha^{3)}$ on the surface of the three-dimensional body V. The condition (4) for different values of the coefficient α contains three types of boundary conditions. Derichlet's boundary conditions mean that on the surface of the body at each moment of time the temperature distribution is specified ($\alpha\to\infty$, $q=0$). Newmann's boundary conditions make the heat flux q through body surface ($\alpha=0$, $q\neq 0$). Newton's boundary conditions formulate the law of heat exchange between the body surface and the environment at the given size Θ($q=0$, $\alpha\neq 0$) .

Let’s consider the case when the cylindrical system of coordinates (α1= r; α2= φ α3= x) is used for the solution. In this case, the Lame parameters are determined like $H_1=H_3=1, H_2=r$. As result the equation (1) will be such:

$$\frac{\partial T}{\partial t} = a\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial T}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 T}{\partial \varphi^2} + \frac{\partial^2 T}{\partial x^2}\right] + \frac{aW_*}{\lambda r}.$$
(5)

Thus, the heat-transfer equation (5) together with the initial conditions (3) and boundary conditions (4) allow us to determine the axially symmetrical temperature field in the cylindrical body, if every time the value $W_\bullet$ is known. For its determination, in the case of cyclic load, it is necessary to know the solution to the corresponding non-stationary problem of thermomechanics.
The main task of the nonstationary theory of thermo-elastic plasticity is the determination of the temperature field of displacements (velocities) and the component of tensors of stresses and strains that occur in the spatial body in the process of its load and heating, when some elements of the body work beyond the limit of elasticity of material. The load process will be considered such that develops over time, which can cause movement of individual body parts. Let isotropic and homogeneous three-dimensional body *V* that is enclosed by surface *S*, in the initial moment of time *t*=0 is in the natural relaxed state at temperature T_0 (α^i), where α^I is axis of arbitrary orthogonal coordinate system, *i* = 1, 2, 3. The body is then subjected to heating and external load forces $\vec{K}(\alpha^i, t)$, which can affect every element of the body and surface forces $\vec{\Sigma}_n(\alpha^i, t)$ that act on the part of body surface S_Σ. The velocities of movements $\vec{V}(\alpha^i, t)$ are set on another part of the body surface S_v, which can be somehow fixed. Let's assume that the heat and load of the body occur so that the resulting deformation can significantly affect the temperature change of the element. We will consider such processes of load and levels of temperatures at which the rheological properties of the material can't be occurred. The configuration of the body is given by the equation of the surface $\Phi(\alpha^i) = 0$, which limits its. In addition, the thermophysical and mechanical properties of the body material and conditions of its heat exchange with the environment must be set. Thermophysical properties of the material are characterized by thermal conductivity λ and thermal diffusivity λ that do not depend on temperature. Heat transfer conditions are set in the form of appropriate boundary conditions, and the mechanical characteristics of the material in the study of deformation processes on straight-line trajectories and trajectories of the lesser curvature are specified as instant charts of tensile specimens, obtained at various fixed temperatures. In addition, values of Poisson's ratio and the linear thermal expansion α_T are also set.

On the basis of these data, it is necessary to determine the temperature T(α_i,t), the three components of the velocity vector of displacements $\upsilon_k(\alpha^i, t)$, the six components of the stress tensor $\sigma_{ij}(\alpha^k, t)$ and the six components of the deformation tensor $\varepsilon_{ij}(\alpha^k, t)$. Here *i, j, k*=1, 2, 3.. Therefore, 16 unknown functions of time and position should be determined. To do this, you'd use the equations of motion, geometrical and physical equations and the heat equation (2.1).

After determining the field of temperature for various time moments the components of the velocity vector of the displacements and the components of the tensors of stress and strain are determining, they should satisfy the three differential equations of motion, six geometric equations and six physical equations. These 15 equations are solved under certain initial and boundary conditions. Initial conditions are specified for all 15 unknown

$$\upsilon_i(\alpha^k) = V^\circ(\alpha^k), \qquad \sigma_{ij}(\alpha^k) = \Sigma^\circ_{ij}(\alpha^k), \qquad \varepsilon_{ij}(\alpha^k) = E^\circ_{ij}(\alpha^k), \tag{6}$$

i, j, k = 1, 2, 3 when t =0.

On the part of the body surface S_Σ, is, where the surface forces Σ_{in} (α^k, t) are set, the components of the stress tensor must satisfy three boundary conditions

$$\Sigma_{in}(\alpha^k, t) = \sigma_{ij} \cdot n_j \quad , \tag{7}$$

where n_j means the directing cosines of the outward norm to the body surface at the corresponding point. On the other part of surface S_V, where components of the velocity vector of the displacements $V_i(\alpha^i, t)$ are set, velocity of movement υ_i should take such value

$$\upsilon_i = V_i(\alpha^k, t). \tag{8}$$

An alternative formulation of the boundary conditions, is when there are three conditions taken in a certain way from conditions (7) and (8) on the surface of the body. Determining 15 unknown can be done in different ways depending on the choice of main unknown. We will use a variant of the mixed method solution, as the main unknown taken three components of the velocity vector of displacements and six components of the stress tensor, for which boundary conditions are formulated. All components of the deformation tensor, which are then determined on the basis of already known components of the velocity vector of displacements, are excluded from six physical equations using geometric Cauchy's relationships.

The equations of motion of an infinitesimal volume element of continuous environment, which is subject to deformation, in the orthogonal coordinate system α^1, α^2, α^3 in the geometrically linear case can be written as

$$\frac{\partial \upsilon_i}{\partial t} = \frac{1}{\rho H_j} \cdot \frac{\partial \sigma_{ij}}{\partial \alpha^j} + B_i(\sigma_{mn}) \tag{9}$$

де i, j, n, m =1, 2, 3, а ρ is density.

In the geometrically linear case, the strain rate can be written

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2}\left(\frac{1}{H_j}\cdot\frac{\partial \upsilon_i}{\partial \alpha^j} + \frac{1}{H_i}\cdot\frac{\partial \upsilon_j}{\partial \alpha^i}\right) + C_{ij}. \qquad (10)$$

Some introduced notations

$$C_{11} = \frac{1}{H_1 H_2}\cdot\frac{\partial H_1}{\partial \alpha^2}\cdot\upsilon_2 + \frac{1}{H_1 H_3}\cdot\frac{\partial H_1}{\partial \alpha^3}\cdot\upsilon_3, \quad C_{22} = \frac{1}{H_2 H_3}\cdot\frac{\partial H_2}{\partial \alpha^3}\cdot\upsilon_3 + \frac{1}{H_2 H_1}\cdot\frac{\partial H_2}{\partial \alpha^1}\cdot\upsilon_1,$$
$$C_{33} = \frac{1}{H_1 H_3}\cdot\frac{\partial H_3}{\partial \alpha^1}\cdot\upsilon_1 + \frac{1}{H_2 H_3}\cdot\frac{\partial H_3}{\partial \alpha^2}\cdot\upsilon_2, \quad C_{12} = -\frac{1}{2H_2 H_1}\cdot\left(\frac{\partial H_2}{\partial \alpha^1}\cdot\upsilon_2 + \cdot\frac{\partial H_1}{\partial \alpha^2}\cdot\upsilon_1\right),$$
$$C_{13} = -\frac{1}{2H_1 H_3}\cdot\left(\frac{\partial H_3}{\partial \alpha^1}\cdot\upsilon_3 + \cdot\frac{\partial H_1}{\partial \alpha^3}\cdot\upsilon_1\right), \quad C_{23} = -\frac{1}{2H_2 H_3}\cdot\left(\frac{\partial H_2}{\partial \alpha^3}\cdot\upsilon_2 + \cdot\frac{\partial H_3}{\partial \alpha^2}\cdot\upsilon_3\right). \qquad (11)$$

The system of equations (9), (10) is closed by physical relationships, connecting stress and strain.

During the solving non-stationary problems of termoplasticity we will use the constitutive equations describing non-isothermal processes of load both on straight-line trajectories, and the trajectories of deformation of the lesser curvature. After solving the problem at the geometry of trajectory of deformation it is possible to discuss the validity of the used constitutive relations.

One of the aspects of general problem of solving non-stationary problems for inelastic bodies is the choice of determining relationships between stresses and strains. This choice is justified by consistency with experiment and is closely related to the studied deformation process. In the general case, the strain values are functions of the change process voltage and temperature, which are determined by the characteristics of all the previous changes in physical factors, not only the current values.

Let's study the physical correlations used for the study of both processes. To do this, we split the load process of the body by time for separate, small enough, stages; the relationship between stresses and deformations of the form are written on each of them using the axiom of isotropy by A. Illyushin and the law of the elastic volume change (4).

$$\sigma_{ij} = 2G^*(\varepsilon_{ij} + \varepsilon_{ij}^{(n)}) + (3\lambda^*\varepsilon - K\varepsilon_T)\delta_{ij}, \qquad (12)$$

where $i, j = 1, 2, 3$, δ_{ij} - Kronecker delta, also

$$\lambda^* = \frac{2G(1+\nu) - 2G^*(1-2\nu)}{3(1-2\nu)}, \quad K = \frac{2G(1+\upsilon)}{1-2\upsilon}, \varepsilon_T = \alpha_T(T - T_0)$$

Here G is the shear modulus, ν is the Poisson's ratio, α_T is coefficient of linear thermal expansion. The values G, ν, α_T in general case are assumed to be temperature-dependent. T G* i $\varepsilon_{ij}^{(n)}$ have different views depending on the used ratio of plasticity: the correlations of the theory of simple load processes or ratio process of the lesser curvature. The specification of the determining equations is reduced to the instant thermomechanical surface. This coversine should be approximated somehow in order to do calculations. Assume that the equation of the instant thermomechanical surface is in the form of a table of experimental data $(\sigma)_i$, $(\varepsilon)_i$, i = 0,1,..., N with fixed T. So let's make the approximation of these data using spline-functions.

The estimated system of equations. The above mentioned complete system of 16 equations (1), (9) (10), (12), is useful for constructing the solution of unsteady associated thermo-elastic and plastic problem for three-dimensional bodies. Let's bring the system to the form

$$\frac{\partial \vec{W}}{\partial t} = \sum_{i=1}^{3} A_i \frac{\partial \vec{W}}{\partial \alpha_i} + \vec{B}, \qquad (13)$$

where $\vec{W}$ is the vector whose components are the needed values

$w_1=v_1$;	$w_4=\sigma_{11}$;	$w_7=\sigma_{12}$;	$w_{10}=\varepsilon_{11}$;	$w_{13}=\varepsilon_{12}$;
$w_2=v_2$;	$w_5=\sigma_{22}$;	$w_8=\sigma_{13}$;	$w_{11}=\varepsilon_{22}$;	$w_{14}=\varepsilon_{13}$;
$w_3=v_3$;	$w_6=\sigma_{33}$	$w_9=\sigma_{23}$;	$w_{12}=\varepsilon_{33}$;	$w_{15}=\varepsilon_{23}.$

To determine the vector $\vec{B}$ and matrix A_i, $i = 1,2,3$ we differentiate the main correlations (12) by time. Thus we assume that G^*, λ^*, K and α_T can be variable values. The result can be written like

$$\frac{\partial \sigma_{ij}}{\partial t} = 2G^* \left(\frac{\partial \varepsilon_{ij}}{\partial t} - \frac{\partial \varepsilon_{ij}^{(n)}}{\partial t}\right) + 3\lambda^* \cdot \frac{\partial \varepsilon_0}{\partial t} \cdot \delta_{ij} + 2 \cdot \frac{\partial G^*}{\partial t} \cdot \acute{Y}_{ij} + b_{ij}, \qquad (14)$$

where
$$b_{ij} = -\left[K \cdot \alpha_t + (T - T_0) \frac{\partial}{\partial t}(K \cdot \alpha_t) \right] \delta_{ij} \cdot \frac{\partial T}{\partial t}. \qquad (15)$$

As result the determining correlations lead to

$$\frac{\partial \sigma_{ij}}{\partial t} = a_{ijkl} \cdot \frac{\partial \varepsilon_{kl}}{\partial t} + b_{ij}, \qquad (16)$$

where
$$a_{ijkl} = 2G^* \delta_{kl} \cdot \delta_{ij} + \lambda^* \delta_{kl} - \left(1 - \frac{\delta_{kl}}{3}\right)(G^* - G_t) \frac{Э_{ij} \cdot Э_{kl}}{\Gamma^2}$$

During the numerical solution of this system it's convenient to lead to a slightly different view. For this we need to eliminate the velocities of deformations in determining relationships using geometric correlations (10). The result can be written

$$\frac{\partial \sigma_{ij}}{\partial t} = \frac{1}{2} a_{ijkl} \left(\frac{1}{H_l} \cdot \frac{\partial v_k}{\partial \alpha^l} + \frac{1}{H_k} \cdot \frac{\partial v_l}{\partial \alpha^k} \right) + d_{ij}, \qquad (17)$$

where $$d_{ij}=\frac{1}{2}a_{ijkl}\cdot c_{kl}+b_i,\ i,j,k,l=1,2,3.$$

Now, comparing the system (17) with the vector equation (13), we define the vector $\vec{B}$ and matrix A_i, $i=1,2,3$. The result can be written

$$\vec{B}=\{B_1,B_2,B_3,d_{11},d_{22},d_{33},d_{12},d_{13},d_{23},c_{11},c_{22},c_{33},c_{12},c_{13},c_{23}\},\tag{18}$$

and non-zero elements of the matrix $(A_i)_{m,s}$ (m – number of line, s - number of column, A_i - matrix 15×15) have the form [3]

$$(A_1)_{1,4}=(A_1)_{2,7}=(A_1)_{3,8}=\frac{1}{\rho H_1},\quad (A_2)_{1,7}=(A_2)_{2,5}=(A_2)_{3,9}=\frac{1}{\rho H_2},$$

$$(A_3)_{1,8}=(A_3)_{2,9}=(A_3)_{3,6}=\frac{1}{\rho H_3},\ (A_1)_{10,1}=\frac{1}{H_1},\ (A_2)_{11,2}=\frac{1}{H_2},\ (A_3)_{12,3}=\frac{1}{H_3},$$

$$(A_1)_{13,2}=(A_1)_{14,3}=\frac{1}{2H_1},\ (A_2)_{13,1}=(A_2)_{15,3}=\frac{1}{2H_2},\ (A_i)_{4,k}=\frac{1}{H_i}\cdot\hat{a}_{11ik},\tag{19}$$

$$(A_i)_{5,k}=\frac{1}{H_i}\cdot\hat{a}_{22ik},\ (A_i)_{7,k}=\frac{1}{H_i}\cdot\hat{a}_{12ik},\ (A_i)_{8,k}=\frac{1}{H_i}\cdot\hat{a}_{13ik},\ (A_i)_{9,k}=\frac{1}{H_i}\cdot\hat{a}_{23ik}.$$

Here $\hat{a}_{ijkl}\equiv\frac{1+\delta_{kl}}{2}\cdot a_{ijkl}$, $l,k=1,2,3$, (summation by l, k isn't done).

Thus, the system of equations (1), (13), required for the solution of unsteady related problems in the theory of thermomechanic is obtained.

The numerical results. Let's consider the field of a cylindrical body (Fig.1). The cylinder is filled with fluid with temperature $T=1536^0$ and with the speed of bleeding through into the body τ.

Denote $T(t,x,r)$ as the temperature field, r is the radius of the cylinder, t is time. Let's note the initial and boundary conditions:

$T=T_0=150^0$, for $r=425$ ($\frac{\partial T}{\partial r}=0$) and for $r=247$ ($\frac{\partial T}{\partial r}=0$), for $x=0$, $x=3110$ ($\frac{\partial T}{\partial r}=0$), where $x\in[0;3110]$, $r\in[247;425]$, $t\in[0;\infty)$.

We need to find the temperature field, the elements of the tensors of stress and strain.

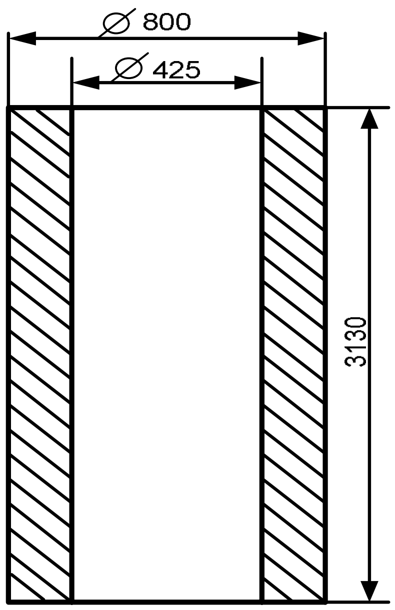

Fig. 1

To find the temperature field of an elastic - plastic body we will use the method of fractional steps (2, 5, 6). Partial derivatives of the coordinates were approximated using the spline functions (1, 3).

Overlay a grid, and calculate its features: divide the length of the cylinder into 70 parts - 3130:44,4 = 70, and a half width into 5 - (425-247)/5 = 44,5. Conduct appropriate notation - - $h = 44{,}5$ мм., $r = 5h$, $x = 70h$.

Replacing the corresponding differential operators in equation (1) of difference (2), let's cheat every node of the grid in two steps - first to the floor step, and then to the full one. Fix the state of the cylindrical body at the time when the temperature of the load has taken the nodes, calculate the temperature field and the rate of change of temperature (4).

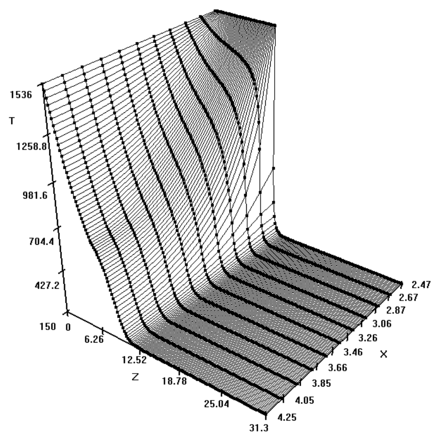

Fig. 2 (100t.bmp - in 100 by time)

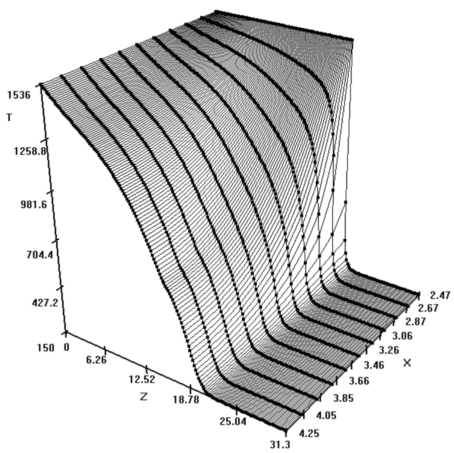

Fig. 2 (200t.bmp - in 200 by time)

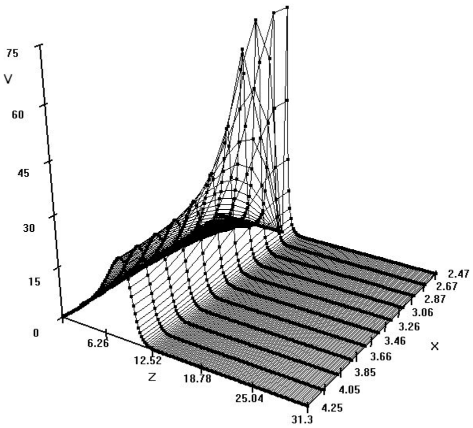

Fig. 3 (100t.bmp - in 100 by time)

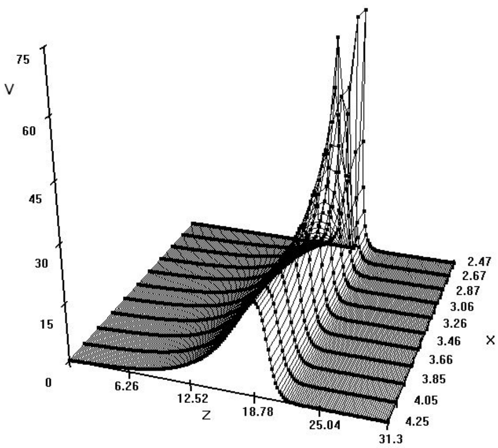

Fig. 3 (200t.bmp - in 200 by time)

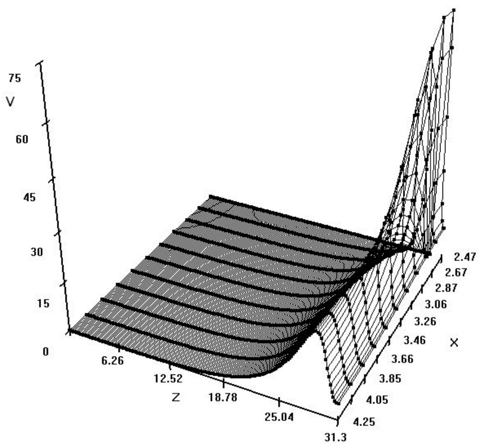

Fig. 3 (200t.bmp - in 200 by time)

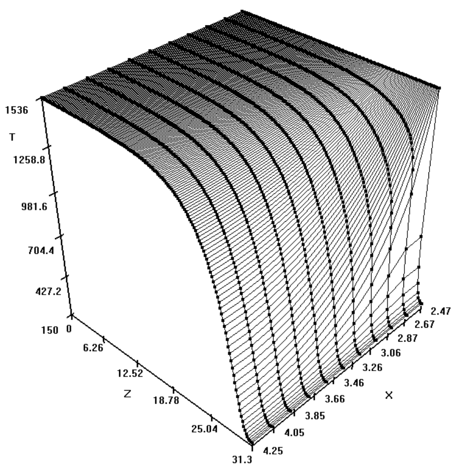

Fig. 4 (300t.bmp - in 300 by time)

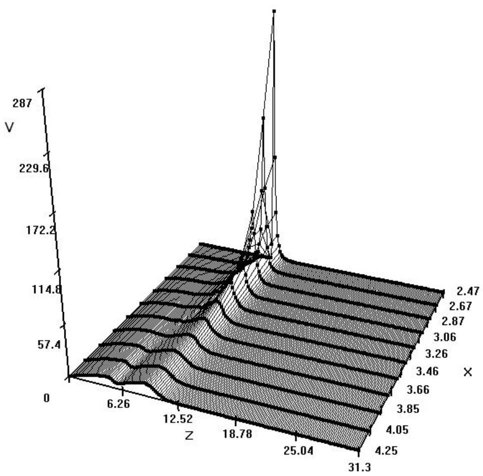

Fig. 5 (100t.bmp - in 100 by time)

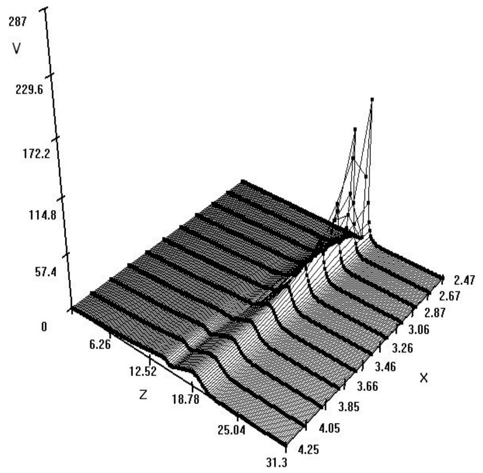

Fig. 6 (200t.bmp - in 200 by time)

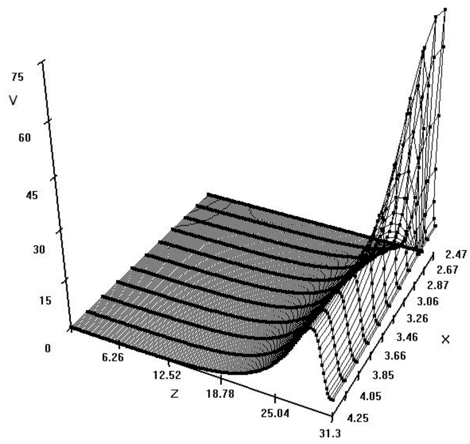

Fig. 7 (300t.bmp - in 300 by time)

Conclusions. The fractional step method for unsteady problems of thermoelasticity for cylindrical bodies is applied; as result the temperature field of the body, the displacement, velocity field of stress and strain are identified. These results have a higher order of accuracy in comparison with similar results obtained using the delta method.

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