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Formulas for the Thin Lenses at Various Orientations of Refracting Surfaces

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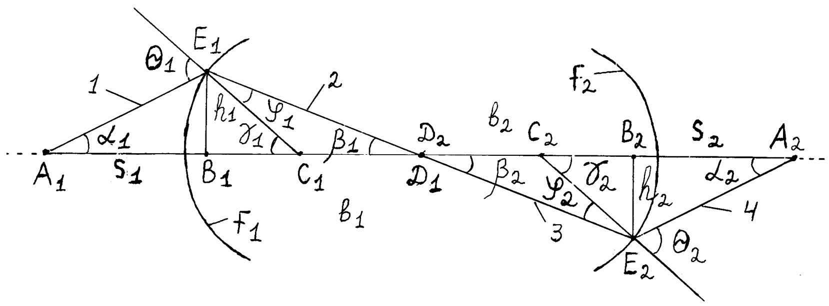
Annotation: *In this article equations for a thin lens are analytically derived, considering that refracting surfaces F1 and F2 can have various orientations with respect to each other. Surface convexities can have various orientations relative to chosen direction of a coordinate axis: left-left, left-right, right-right, right-left. It is established that mutual orientation of surfaces can increase or decrease absolute value of focal length of a thin lens. It is shown that use of absolute values of geometrical parameters makes it easier to derive equations.*

Introduction

It is known that the equation of a thin lens has wide application in engineering practice (1, 2)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad (1)$$

where f – focal length, $n = n_2/n_1$ - a relative refraction index of medium, n_1 and n_2 - refraction indices of medium 1 and 2 , R_1, R_2 – radii of spherical surfaces of F_1 and F_2, figure 1. Interest to thin lenses increases regarding their use in fiber-optical sensors and in fiber-optical technologies (3,4). Thin lenses are widely used in instrumentation, medicine and various branches of the light and heavy industry. In order to determine refraction paths of beams in figure 1 the lens of finite thickness d is represented.



$A_1B_1 = s_1$, $A_2B_2 = s_2$, $B_1D_1 = b_1$, $B_2D_2 = b_2$, $R_1 = C_1E_1$, $R_2 = C_2E_2$

Figure 1 – Scheme of beam passing through a lens with finite thickness

Theoretical analysis and solution

Rather simple derivation of equation for a thin lens demanded the detailed analysis and revision for following reasons. There are two approaches for derivation of the equation for a thin lens in references. In the first approach direct and simultaneous application of the refraction law to both conditionally flat refracting surfaces F_1 and F_2 is used (5-7), shown in Figure 1:

$$\frac{n_1}{s_1} + \frac{n_2}{b_1} = (n_2 - n_1)\left(\frac{1}{R_1}\right), \qquad (2)$$

$$\frac{n_1}{s_2} + \frac{n_2}{b_2} = (n_2 - n_1)\left(\frac{1}{R_2}\right). \qquad (3)$$

We will prove that it is wrong to interchange locations of n_1 and n_2 at the right side of the equation (3) as it is done in (5-7) for the beam leaving a lens. Let's consider Figure 1 to prove this statement. The travel direction of light beam shouldn't influence on values of scalar quantities, such as angles: $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2, \gamma_1, \gamma_2$ and on values of their trigonometric functions. Therefore physical situations in case if light beam travels in the direction A1, E1, D1, D2, E2, A2, and in case if it travels in reverse direction will be identical. This statement is proved based on following reasoning: a triangle D2, E2, A2 on the surface F2 cannot have negative value for the left side of the equation (3) for the beam passing through a lens of finite thickness d, Figure 1. In this case we only consider absolute values of geometric quantities. As a result, it is established that the direction of beam travel doesn't have any influence on result.

After solving system of equations (2)-(3), we get the following result for a lens of finite thickness d, in which the assumption is taken as $b_2 = b_1$ $(if\ R_2 = R_1)$,

$$\frac{1}{s_1} - \frac{1}{s_2} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) . \qquad (4)$$

This equation differs from earlier known results: on the left side of the equation 4 there is subtraction (instead of addition) of absolute values of scalar quantities. In the equation (4) s_2 is taken as scalar, therefore its absolute value is used. It follows from Figure 1, where all quantities are used as scalars (their absolute values are used), without consideration of change of linear and angular directions relative to some coordinate system.

If solution of the system (2)-(3) is found using the condition $b_2 = -b_1$, as it was done in [5] on p. 119-120, then equation (4) will have the form below:

$$\frac{1}{s_1} + \frac{1}{s_2} = (n - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \qquad (5)$$

This equation (5) differs from result obtained in [5], because in the equation (5) we didn't interchange $n_2 \rightleftarrows n_1$.

However, this expression is not a formula for the focal length, since as value $s_1 \to \infty$ approaches infinity, value $s_2 \to \infty$ also approaches infinity. Also in case when $R_1 = R_2$, s_1 is equal to s_2. This is a consequence of finite thickness of a lens, Figure 1. Replacement $b_2 = -b_1$ in this case doesn't have physical justification and artificial change of the sign is incorrect. This confirms the previously known assertion that the concept of focal length makes sense only for a thin lens. The equations (6) and (7), which we will prove exactly later, in (5) are derived from the incorrect and unjustified replacement $b_2 = -b_1$ and artificial swapping of locations of the two refractive indices $n_2 \rightleftarrows n_1$.

Similarly in the equation (5) of the paragraph 318 (6) the minus sign before the member $\frac{1}{R_2}$ occurs as a result of replacing two mediums: medium with a value of index n_1 is replaced by medium with value of index n_2. Therefore there is a binding to the beam, which travels from a lens to air through surface F_2. Also without correct justification it is taken that $s_2 < 0$ in the equation (3) of the paragraph 318-[6], though it is located in the right half-plane in this coordinate system. As a result correct equation (5) of the paragraph 318 (6) is actually received on the basis of casual coincidence. Here left-left orientation of refracting surfaces is considered. Artificial change of the sign for s_2 incidentally led to the correct result for a thin lens equation with the left-left orientation, see Figure 247 and the equation (5) of the paragraph 318(6). For the proof of this statement we will show that, when carrying out the same derivation according to equations (1) - (5) from the paragraph 318 (6) for a lens with the left-right orientation, the same equation (5) from the paragraph 318 (6) will be obtained, in which the member $\frac{1}{R_2}$ has a minus sign. However in this case (for the left-right orientation of surfaces F_1 and F_2) this term has to have a plus sign. Therefore equation (1) needs to be valid. It follows that solution of the system of equations (2)-(3) for two surfaces of a thin lens doesn't lead to correct result. By this we mean absence of the detailed consideration of laws of refraction and of change of direction of beams on both sides of two various refracting surfaces F_1 and F_2. The same comments can be made about results in (5). Artificial replacement of two indexes of refraction $n_2 \rightleftarrows n_1$ in reality leads to violation of the laws of refraction. In particular, it leads to violation of the Snell's law on the second refracting surface. In this case signs of the terms $\frac{1}{R_2}$ and $\frac{1}{s_2}$ will be incorrect.

Equation (4), from the point of view of practical use, is less informative, though it is formally right for a lens with finite thickness d. In other words, the solution of the system of equations, when b_2 is expressed through b_1 ($b_2 = \pm b_1$) and when artificial replacement $n_2 \rightleftarrows n_1$ is done, doesn't lead to useful and correct result. Especially this approach can't be applied (or directly extended) to a thin lens. This formal transition from properties of a lens with finite thickness to a thin lens doesn't have physical justification. The equations (4)-(5), which apply for a lens with finite thickness, are interesting, because they contain two members of internal focal lengths: $f_1 = \frac{1}{(n-1)\frac{1}{R_1}}$, $f_2 = \frac{1}{(n-1)\frac{1}{R_2}}$.

The second approach leads to correct and full result. The second approach is based on similarity of properties of a thin lens and of a triangular prism (1,2):

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}, \qquad (6)$$

where $\frac{1}{f}$ corresponds precisely to equation (1). Therefore the detailed analysis of beam travel path is taken into account and refraction laws on surfaces F_1 and F_2 are considered. However equations (1) and (6) apply for a thin lens with oppositely oriented convexities of surfaces F_1 and F_2: to the left and to the right from the center of coordinate system, Figure 195, paragraph 88 (1). Then a question arises: how to correctly derive equation for a thin lens, which has both surfaces of F_1 and F_2 with the same orientation of convexities, Figure 2? In other words, it is necessary to theoretically and correctly obtain the following equation:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \qquad (7)$$

which describes a lens with the same orientation of convexities of two refracting surfaces: left-left (or right-right) orientation, Figure 2.

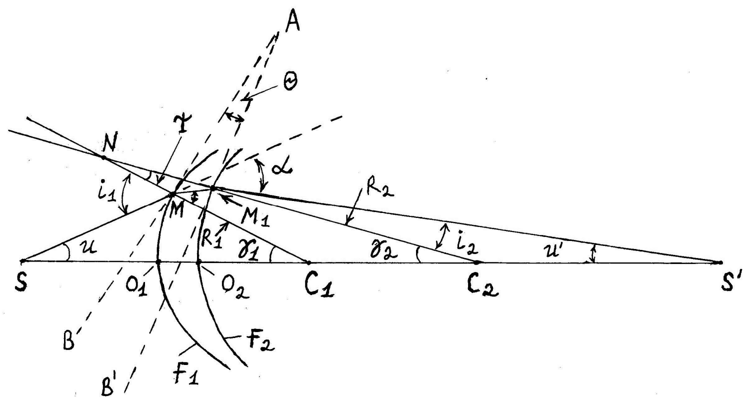


Figure 2 – Scheme of a thin lens. Convexities of surfaces F_1 and F_2 have the same left-left orientation.

The conclusions given in (5-7) don't allow us to show correctly (theoretically) why in one case the equation (1) works, and why in the other case equation (7) works. Presence of a sign (+) or (-), before the member $(1/R_2)$ in the equation for focal length of a thin lens, shouldn't depend on conditionally accepted principle of dependence of a sign on the direction of beam travel. Presence of a sign has to follow theoretically from a concrete physical law and from its corresponding mathematical description. In other words, simple replacement of one environment by another shouldn't play any role on the result. Here we are talking about replacement of n_1 with n_2 for the second half-plane, when $x_2 > 0$. Also the result shouldn't depend on the sign of geometrical quantities in any coordinate frame.

We established that use of analogy between a prism and a thin lens, used in (1, 2), leads to the full and correct solution of the given task. For the proof of it let's see Figure 2. In this case surfaces F_1 and F_2 have convexities oriented in one direction. In some references the following terms can be used: convex-concave, concave-convex surfaces, doubly convex and doubly concave surfaces. However, on our opinion, these terms are not clear at describing different orientations of surfaces. We believe that it is preferable to use the following terms: left-left, right-right, left-right and right-left orientations of surfaces F_1 and F_2. This notation uses location of surfaces relative to the center of optical system, which in the case coincides with the center of coordinate system O.

We didn't see in literature references available to us analytical analysis of such orientation of surfaces (with the same orientation of convexities). We tried to adhere to notations, which are used in Figures 195, 197 of paragraphs 88,89 (1), in Figure 247 of paragraph 318 (6) and in Figure without numbering on the page 119 (5). This was done in order to easily see similarities and differences of our solution from theirs.

From Figure 2 it follows that $\gamma_1 = \Psi + \gamma_2$. This in turn leads to

$$\Psi = \gamma_1 - \gamma_2, \tag{8}$$

$$\alpha = u + u', \tag{9}$$

$$\Psi = \theta . \tag{10}$$

Equations (8-10) are important to receive further results. Our subsequent calculations are similar to the ones carried out in paragraphs 88 and 89 (1). They allow us to accurately receive the equation (7) for the left-left orientation of surfaces. Without giving detailed analysis as in (1), here we will only note that the derivation consists of two parts. In the first part, Figure 195, paragraph 88(1), equation (1) is proved. In the second part, Figure 197, paragraph 89(1), the equation (5) is proved. During the derivation the following approximations were used (1): $u \approx \sin u = h_1/s_1$, $u' \approx \sin u' = h_2/s_2$, $\gamma_1 \approx \sin\gamma_1 = h_1/R_1$, $\gamma_2 \approx \sin\gamma_2 = h_2/R_2$, $\varphi \approx \sin\varphi = h/f$, $h_2 \approx h_1 \approx h$.

Thus analytically it was proven that, when convexities of surfaces of a thin lens have different orientation (the left-right orientation), then equation (1) is valid. When surfaces have the same orientation (the left-left orientation) then equation (7) works. This is explained by dependence of angles Ψ, θ, γ_1 , γ_2, u, u', α from each other. When orientations of surfaces F_1 and F_2 are opposite (the left-right orientation), then equation (8) will have the following form (1, 2):

$$\Psi = \gamma_1 + \gamma_2 . \tag{11}$$

In Figure 3 opposite orientation of surfaces F_1 and F_2 (right-left orientation) is shown.

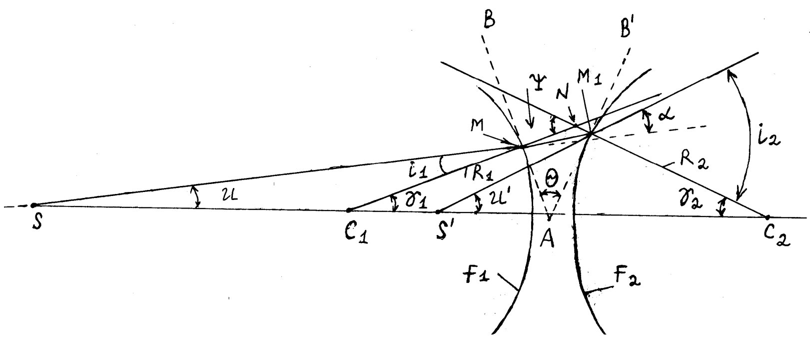
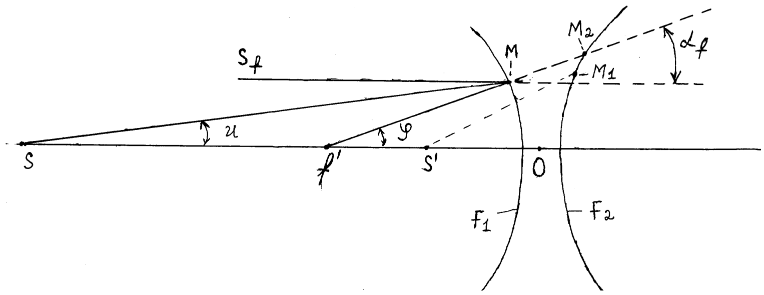


Figure 3 - Scheme of a thin lens. Convexities of surfaces F_1 and F_2 have opposite orientation.

In references available to us such orientation of surfaces isn't considered. Analysis shows that in this case, calculations on the basis of a thin prism approach lead to the correct results. Namely, equations (10)-(11) are valid. In Figure 3 $C_1M=R_1$, $C_2M' = R_2$-radii of surfaces. Triangle BAB' in this case has different orientation, than shown in Figure 2.

The only difference will be in S', which will be the imaginary image of S. Therefore the focal length for such right-left lens will have negative value (we note that it is convention only), which can be observed from Figure 4.



S_f - beam from infinitely remote object, f' - imaginary focus.

Figure 4 – Scheme of refraction of beam from a remote source

From Figures 3 and 4 it can be seen that $\varphi = \alpha_f = \alpha$. It is proved by a similar way as in paragraph 89(1): on the basis of proximity of points $M_1$and M_2 on the surface F_2, Figure 4. Interesting difference of this lens is that:

$$\alpha = u' - u\,. \qquad (12)$$

For the angle Ψ the equation (11) is still valid.

Based on equations (11) and (12) and using identity $\alpha = (n-1)\theta$, which is proved in (1, 5), the equation for a thin lens can be found (Figures 3 and 4):

$$\frac{1}{s_2} - \frac{1}{s_1} = \frac{1}{f'} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \qquad (13)$$

$s_2 = OS'$, $s_1 = OS$, O – the central point between surfaces of F_1 and F_2, Figure 4. In the equation (13) all quantities are scalars with absolute values. Therefore, on the left side of (13) there is a simple difference of two positive quantities $\frac{1}{s_2}$ и $\frac{1}{s_1}$.

The rule, accepted in geometrical optics, of defining a sign of physical and geometrical quantities depending on their arrangement in a certain coordinate system (for measurement of point coordinates and of the directions of angle changes) in certain cases complicates the proof of simple results. It can sometimes lead to purely mechanical mistakes. In (6) actually positive value s_1 is written in equation with a minus sign conditionally, because it is located on the negative (left) half-plane in the chosen system of coordinates. If equation (2) is used for F_2 surface instead of equation (3), then quantity s_2 is used with a minus sign as the imaginary image, in spite of the fact that at present it is located on the right positive half-plane, Figure 247 of the paragraph 318 (6), (7). Therefore in some special cases in engineering practice, in order to not have wrong signs of required physical quantities, it is easier and more reliable to work with absolute values of distances and angles, as it is used in Figures 1-4 and in derivation of equations (2)-(13) for a lens of finite thickness d and for a thin lens. In (1, 2, 9) scalar representation of physical quantities is used in showing results of geometrical optics, therefore calculations are conducted only with absolute values of physical and geometrical variables.

Consideration of vectors, of their directions and of their rotations justifies itself in mechanics. For example, vectors are useful when considering laws of conservation of momentum and of angular momentum in a certain coordinate system.

Detailed consideration of theory of a thin lens in some courses of the general physics for institutions of higher education can be absent (8). Probably it is considered that this material is rather elementary. Also in courses of general physics for schools we didn't meet rather detailed statement of this material. Only in sources (1) and (5) some detailed summary of the material is presented. Therefore we hope that our article will fill this gap and will be useful to engineers and the production workers who are engaged in development and design of new optical devices.

The thin lens isn't simple as it seems at the first sight. In (9) equation (5) is presented in the form of equation (4) only on the basis of conditional change of the sign, which is related to a certain coordinate system, figure 120, paragraph 50. The center of the coordinate system is connected with the optical center. The minus sign before the member $\left(\frac{1}{R_2}\right)$ in this case appears (unreasonably from the physics point of view) as a result of the location of the center of curvature of a surface F_2 on the left half-plane (equation 12.15 of the paragraph 50 (9)). It is purely mechanical result, a convention, which doesn't facilitate perception of simple truth.

On the base of proof presented in (9), we established that it is possible to justify correctness of equations (5) and (6) for the left-left orientation of surfaces F_1 and F_2. In paragraph 50 (9) only left-right orientation is considered. This is classical orientation. Equality of the sum of angles is taken as a basis $i_1' + i_2' = \gamma_1 + \gamma_2$, see figure 120 and a formula (12.7) of (9). The difference between conclusions in (9) from conclusions in (1) is that properties of the angle θ aren't used at vertex A of triangular prism, see Figures 2 and 3.

In our case (Figure 2, the left-left orientation of surfaces) we receive the identites: $\Psi = i_1' - i_2' = \gamma_1 - \gamma_2$, $i_2 = \gamma_2 - u'$, $i_1' = (u + \gamma_1)/n$. Further calculations correspond to the transformations which have been presented in paragraph 50-[9]. $< NM_1M = i_2'$, $< C_1MM_1 = i_1'$. Sine and tangents of small angles are replaced with approximate formulas. In this part, designation of some angles corresponds to those used in (9). Therefore, for this case also equations (5) and (6) are proved analytically.

Thus, we have established that signs before members of the derived equations are defined not by an arrangement of points of S, S', C_1, C_2 on the left (negative) or right (positive) half-plane, relative to origin of the coordinate system, Figure 2. Everything depends on relationships between angles: $\Psi, \theta, i_1, i_1', i_2, i_2', u, u', \gamma_1, \gamma_2$, which are formed at various orientations of refracting surfaces F_1 and F_2.

We are considering absolute values of variables. It should be noted that using properties of a thin prism and use of properties of the angle θ (its relation with other angles) at A vertex somewhat facilitates the proof. Unlike in situations when, by considering a geometrical figure

– a triangle of MNM_1 and by considering internal and external angles, formed by this figure, direct relationships between angles $\Psi, \theta, i_1, i'_1, i_2, i'_2, u, u', \gamma_1, \gamma_2$ is obtained.

Therefore, summing up, in the case of the left-right orientation of refracting surfaces F_1 and F_2, formulas of a thin lens are shown below:

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f} \ , \qquad (14)$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) . \qquad (15)$$

In case of left-left (Figure 2) and the right-right orientation of surfaces F_1 and F_2, formula of a thin lens becomes as shown below:

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f} , \qquad (16)$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) . \qquad (17)$$

In case of the right-left orientation of surfaces (Figure 3), equation of a thin lens have a form as below:

$$\frac{1}{s_2} - \frac{1}{s_1} = \frac{1}{f'} \ , \qquad (18)$$

$$\frac{1}{f'} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) . \qquad (19)$$

Along with the areas stated above, equations of a thin lens could be used in ultrasonic equipment as well (10). For example, the equation (209) of paragraph 2, chapter 7 (10) has similar type with equation given in (1). The difference is existence of some coefficients, which are dependent on the speed of ultrasound and physical characteristics of the environment.

Conclusions

1. Equations for a thin lens are analytically derived, taking into account that refracting surfaces F_1 and F_2 can have various mutual orientations relative to each other. Surface convexities can be directed according to the direction of a coordinate axis: left-left, left-right, right-right, right-left.

2. Mutual orientation of surfaces relative to each other can both increase or decrease absolute value of focal length of a thin lens f.

3. Interdependence of the values f', s_2, s_1 among each other depends on mutual orientation of refracting surfaces F_1 and F_2.

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